Q 10. State Leibniz rule for definite integrals when limits of integration are functions and apply it to find $F'(x)$ given $F(x) = \int_1^{2x} \sqrt{t^2 + t}\ dt$

Solution

Leibniz rule

Given $y = f(t),\ a \le t \le b,$

$$\frac{d}{dx}\int_{v(x)}^{u(x)} f(t)\ dt = \underbrace{f\big(u(x)\big)\{u'(x)\}}_{\substack{I=F(b)\\ Neglect\ when\\ u'(x)=0}} - \underbrace{f\big(v(x)\big)\{v'(x)\}}_{\substack{II=F(a)\\ Neglect\ when\\ v'(x)=0}} \quad (1)$$

Note that variable under definite integral is always a dummy variable. Thus, one can easily replace integrand $f(t)$ by $f(x)$ without making any difference to the problem or to its solution. We chose to denote integrand as $f(t)$ instead of $f(x)$ to avoid unnecessary confusion.

Application

$$F(x) = \int_1^{2x} \sqrt{t^2 + t}\ dt \quad (2)$$

Given	Find
$u(x) = 2x$	$u'(x) = 2$
$v(x) = 1$	$v'(x) = 0$ hence II term on RHS of (1) does not need additional consideration
$f(t) = \sqrt{2t^2 + t}$	$f\big(u(x)\big) = f(2x) = \sqrt{(2x)^2 + (2x)} = \sqrt{4x^2 + 2x}$

Therefore,

$$F'(x) = \frac{d}{dx}F(x) = f\big(u(x)\big)\{u'(x)\}$$
$$= \sqrt{4x^2 + 2x}\,(2)$$
$$= 2\sqrt{4x^2 + 2x}$$

<u>**Answer**</u>