Example (Newton's Law of Cooling)

A beaker of certain liquid initially at 350 °F is suddenly allowed to cool in a large room maintained at 75 °F. If it is known that the temperature of the liquid is changing at the rate of $-110e^{-0.4\,t}\,\frac{°F}{\text{min}}$, what is the temperature of the liquid at the end of 5 minutes of cooling.

Solution 1 - No knowledge of Newton's law of cooling assumed

The rate of cooling is given as $-110e^{-0.4\,t}\,\frac{°F}{\text{min}}$

Let the temperature at any time t be $u(t)$

Therefore,

$$\frac{du}{dt} = -110e^{-0.4\,t}$$

$$\Rightarrow \quad du = \left(-110e^{-0.4\,t}\right)dt$$

$$\Rightarrow \quad \int du\,\text{dt} = \int\left(-110e^{-0.4\,t}\right)dt$$

$$\Rightarrow \quad u(t) = (-110) \times \frac{1}{(-0.4)}\,e^{-0.4\,t} + C$$

$$= 275\,e^{-0.4\,t} + C$$

When $t = 0$, $u(0) = 350 = (275 \times 1) + C$

$$\Rightarrow \quad C = 75$$

Thus, the equation of cooling is:

$$u(t) = 275\,e^{-0.4\,t} + 75$$

Hence,

$$u(5) = 275\,e^{-0.4 \times 5} + 75$$

$$= \frac{275}{e^2} + 75$$

$$= 112.22\ °F$$

$$\approx 112\ °F$$

Solution 2 -Knowledge of Newton's law of cooling assumed

According to Newton's law of cooling

$$\frac{du}{dt} = k\,(u(t) - 75)$$

But the rate of cooling is given as $-110e^{-0.4\,t}\ \frac{°F}{\min}$

Therefore,

$$k\,(u(t) - 75) = -110e^{-0.4\,t}$$

$$u(t) = -\frac{1}{k}\,110e^{-0.4\,t} + 75 \quad \text{-------------------- (1)}$$

But

$$u(0) = 350$$

$$= -\left(\frac{1}{k}\,110 \times e^{0}\right) + 75$$

$$\Rightarrow k = \frac{2}{5}$$

Thus, the temperature function $u(t)$ in (1) is given as

$$u(t) = -\frac{5}{2}\,110e^{-0.4\,t} + 75$$

$$= 275\,e^{-0.4\,t} + 75$$

Therefore,

$u(5) = 275\,e^{-0.4 \times 5} + 75 \approx 112\ °F$ as before.