**Q 31.** Find $\lim_{h\to 0} \frac{(a+h)^2 \sin(a+h) - a^2 \sin a}{h}$

**(IIT JEE)**

**Solution**

Note that simply subtituting $h = 0$ gives $\frac{0}{0}$ so some careful evaluation is needed.

Consider $f(h)$ where

$$f(h) = \frac{(a+h)^2 \sin(a+h) - a^2 \sin a}{h}$$

$$= \frac{(a^2+2ah+h^2) \sin(a+h) - a^2 \sin a}{h}$$

$$= \frac{a^2 \sin(a+h) + 2ah \sin(a+h) + h^2 \sin(a+h) - a^2 \sin a}{h}$$

We look for an arrangement where $h$ in denominator either cancels with some $h$ in numerator or becomes part of $\lim_{x\to 0} \frac{\sin x}{x} = 1$ formula.

Hence,

$$f(h) = \frac{\overbrace{a^2 \sin(a+h) - a^2 \sin a}^{Group\ these} + \overbrace{2ah \sin(a+h) + h^2 \sin(a+h)}^{Group\ these}}{h}$$

$$= \frac{a^2(\sin(a+h) - \sin a)}{h} + \frac{h \sin(a+h)(2a+h)}{h}$$

$$= \frac{a^2 \overbrace{(\sin(a+h) - \sin a)}^{Recall\ (\sin C - \sin D)}}{h} + \sin(a+h)(2a+h) \text{ since } h \neq 0, h \text{ could be canceled}$$

$$= \frac{a^2\{2\cos\left(\frac{a+h+a}{2}\right)\sin\left(\frac{a+h-a}{2}\right)\}}{h} + \sin(a+h)(2a+h)$$

$$= \frac{a^2 \cos\left(a+\frac{h}{2}\right)\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} + \sin(a+h)(2a+h)$$

Hence

$$\lim_{h\to 0} f(h) = \lim_{\frac{h}{2}\to 0} \frac{a^2 \cos\left(a+\frac{h}{2}\right)\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} + \lim_{h\to 0} \sin(a+h)(2a+h)$$

$$= a^2 \lim_{\frac{h}{2}\to 0} \cos\left(a+\frac{h}{2}\right) \times \lim_{\frac{h}{2}\to 0} \frac{\sin\left(\frac{h}{2}\right)}{\frac{h}{2}} + \lim_{h\to 0} \sin(a+h)(2a+h)$$

$$= a^2 \cos a + 2a \sin a$$

**Ans.**