Example

Prove that $\sqrt{2}+\sqrt{3}+\sqrt{5}$ is an irrational number.

Proof

Let $x=\sqrt{2}+\sqrt{3}+\sqrt{5}$ and if possible let $x=\dfrac{p}{q}$ where p and q are non-zero relative primes.

$x-\sqrt{5}=\sqrt{3}+\sqrt{2}$

Squaring both sides succesively after simplification

$x^2-2x\sqrt{5}+5=5+2\sqrt{6}$

$x^4-4x^3\sqrt{5}+20x^2=24$

$x^4+20x^2-24=4x^3\sqrt{5}$

$x^8+40x^6+352x^4-960x^2+576=80x^6$

$x^8-40x^6+352x^4-960x^2+576=0$ --------------------------(1)

Eqn (1) is a polynomial equation of the form $a_0x^n+a_1x^{n-1}+...+a_n=0$ in x with $a_0, a_1, a_2...a_n$ as integer coefficients where $a_0=1$ and $a_n=576$.

Substitute, $x=\dfrac{p}{q}$ as a rational root

$$\frac{p^8}{q^8}-\frac{40p^6}{q^6}+\frac{352p^4}{q^4}-\frac{960p^2}{q^2}+576=0$$

$p^8-40p^6q^2+352p^4q^4-960p^2q^6+576q^8=0$ --------------- **(1a)**

Skip the dimmed text below that was published earlier as it doesn't allow to prove that p is not a factor of 576. Thus for the term $\dfrac{576q^8}{p}$ to be an integer $\dfrac{q^8}{p}$ and 576 do not need to be integers individually.

Divide both sides by $p\neq 0$

$$p^7-40p^5q^2+352p^3q^4-960pq^6+\frac{576q^8}{p}=0$$

$$p^7 - 40p^5q^2 + 352p^3q^4 - 960pq^6 = -\frac{576q^8}{p} \quad \text{----------------------(2)}$$

Since the LHS of above equation is an integer, the RHS $\frac{576q^8}{p}$ must also be an integer. 576 is a known integer therefore, $\frac{q^8}{p}$ must be an integer.

However, $\frac{q^8}{p}$ cannot be an integer since p and q are relative primes.

Eqn (1) has resulted in an inconsistent equation (2) due to the assumption that $x = \frac{p}{q}$ is a rational root. Thus, $\frac{p}{q} = \sqrt{2} + \sqrt{3} + \sqrt{5}$ cannot be a rational number.

Divide both sides of eqn (1a) by $q \neq 0$

$$\frac{p^8}{q} - 40qp^6 + 352q^3p^4 - 960q^5p^2 + 576q^7 = 0$$

$$-40qp^6 + 352q^3p^4 - 960q^5p^2 + 576q^7 = -\frac{p^8}{q} \quad \text{----------------------(2)}$$

Since the LHS of above equation is an integer, the RHS $\frac{p^8}{q}$ must also be an integer. However, $\frac{p^8}{q}$ cannot be an integer since p and q are relatively prime.

Eqn (1) has resulted in an inconsistent equation (2) due to the assumption that $x = \frac{p}{q}$ is a rational root. Thus, $\frac{p}{q} = \sqrt{2} + \sqrt{3} + \sqrt{5}$ cannot be a rational number.