Q Evaluate $\lim_{x\to 0} x^2 \left(1+2+3+\cdots+\left[\frac{1}{|x|}\right]\right)$

Solution

Le $f(x) = [x]$, where $[x]$ is the greatest integer $\leq x$

Let $x = \frac{1}{n}$

Hence

$$\left[\frac{1}{|x|}\right] = [n] = n \in \mathbb{N} \text{ and as } x \to 0,\ n \to \infty$$

Therefore,

$$\lim_{x\to 0} x^2 \left(1+2+3+\cdots+\left[\frac{1}{|x|}\right]\right)$$

$$= \lim_{n\to\infty} \left(\frac{1}{n}\right)^2 (1+2+3+\cdots+[n])$$

$$= \lim_{n\to\infty} \left(\frac{1}{n}\right)^2 (1+2+3+\cdots+n) \text{ as } n \in \mathbb{N}, [n] = n$$

$$= \lim_{n\to\infty} \left(\frac{1}{n}\right)^2 \left(\frac{n(1+n)}{2}\right) \text{ Recall } 1+2+3+\cdots+n = \frac{n(n+1)}{2}$$

$$= \lim_{n\to\infty} \left(\frac{(1+n)}{2n}\right)$$

$$= \lim_{n\to\infty} \left(\frac{1}{2n} + \frac{1}{2}\right)$$

$$= \frac{1}{2} \quad \text{since } \frac{1}{2n} \to 0\ as\ n \to \infty$$

Answer