Q 1. Find one pair of positive integers a, b such that $ab(a+b)$ is not divisible by 7, but $(a+b)^7 - a^7 - b^7$ is divisible by 7^7.

[Problem Source: Harvard Math Dept. Dec 19, 2010
http://www.math.harvard.edu/putnam/index.html]

Solution

We learned to write binomial coefficients of $(a+b)^n$ particularly while $n < 10$. (See video clip on Pascal triangle). It is also possible to expand $(a+b)^7$ without explicit knowledge of Pascal triangle provided one is careful with computation in expanding $(a+b)^7$ written as $(a+b)^3(a+b)^2(a+b)^2$. However, we will use Pascal triangle to obtain (1) as below.

Thus

$$(a+b)^7 = a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7 \quad (1)$$

$\Rightarrow$ $$(a+b)^7 - a^7 - b^7 = 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6$$

$\Rightarrow$ $$LHS = 7ab(a^5 + \underbrace{3a^4b + 5a^3b^2 + 5a^2b^3 + 3ab^4}_{\text{Split each term s.t. factor (a+b) can be extracted. Why (a+b)?}} + b^5)$$

$\Rightarrow$ $$LHS = 7ab(\underbrace{a^5 + a^4b}_{\text{factorize}} + \underbrace{2a^4b + 2a^3b^2} + \underbrace{3a^3b^2 + 3a^2b^3} + \underbrace{2a^2b^3 + 2ab^4} + \underbrace{ab^4 + b^5})$$

$\Rightarrow$ $$LHS = 7ab\{a^4(a+b) + 2b^3a(a+b) + 3b^2a^2(a+b) + 2ba^3(a+b) + b^4(a+b)\}$$

$\Rightarrow$ $$LHS = 7ab(a+b)(a^4 + 2ab^3 + \underbrace{3a^2b^2}_{\text{Split as } 2a^2b^2 + a^2b^2\text{. See recall box}} + 2a^3b + b^4)$$

Recall:
$(a+b+c)^2$
$= a^2 + b^2 + c^2$
$+ 2ab + 2bc$
$+ 2ca$

$\Rightarrow$ $$LHS = 7ab(a+b)(a^4 + 2ab^3 + 2a^2b^2 + a^2b^2 + 2a^3b + b^4)$$

$\Rightarrow$ $$LHS = 7ab(a+b)(a^4 + a^2b^2 + b^4 + 2ab^3 + 2a^2b^2 + 2a^3b)$$

$\Rightarrow$ $$LHS = 7ab(a+b)(a^2 + ab + b^2)^2 \quad (2)$$

It is given that $ab(a+b)$ is not divisible by 7, but the entire RHS of (2) is is divisible by 7^7.

$\Rightarrow$ $7(a^2 + ab + b^2)^2$ must be divisible by 7^7

$\Rightarrow$ $\frac{7(a^2+ab+b^2)^2}{7^7} = \frac{(a^2+ab+b^2)^2}{7^6} = n$ where n is some multiple of 7^6

$\Rightarrow$ $(a^2 + ab + b^2)^2 = 7^6$ for first multiple when $n = 1$

$\Rightarrow$ $(a^2 + ab + b^2)^2 = (7^3)^2$

$\Rightarrow$ $(a^2 + ab + b^2) = 7^3 = 343$ since a and b are positive integers

$\Rightarrow$ $(a^2 + 2ab + b^2) - ab = 343$

$\Rightarrow$ $(a + b)^2 - ab = 343$ (3)

$\Rightarrow$ We are looking for a pair of positive integers a, b such that $(a + b) <$ $(20 = \sqrt{400})$ (Note $\sqrt{343} < 19$)

$\Rightarrow$ By inspection $a = 18$ and $b = 1$ are found

Answer