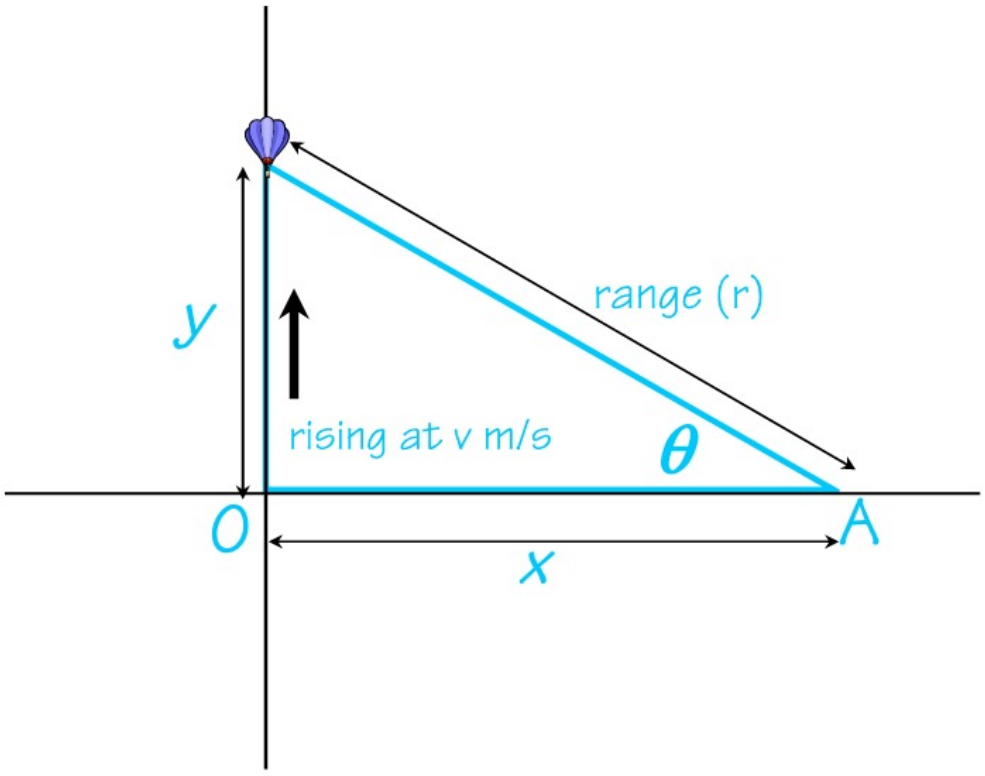
## Example

As shown in the figure, a helium balloon is rising vertically above point O on the ground at a constant rate. A rangefinder at point A tracks the rising balloon and determines its range *r*. If the distance *OA* = *x* m, find equations for the rate at which the angle θ and the range *r* are changing with time when the balloon is *y* m above the ground. Justify units of the rate quantities.

## Solution




θ and *y* change with time, thus $\theta(t)$ and $y(t)$ are functions of *t*. From figure, $\frac{y}{x} = \tan(\theta)$

Therefore,

$$\sec^2(\theta)\left(\frac{d\theta}{dt}\right) = \frac{1}{x}\left(\frac{dy}{dt}\right)$$

$$\frac{d\theta}{dt} = \frac{1}{x\sec^2(\theta)}\left(\frac{dy}{dt}\right) \qquad \textbf{(1)}$$

Eqn (1) is the required equation for determination of rate of change of angle θ with time.

To find units of $\frac{d\theta}{dt}$ consider LHS and RHS of (1) which must be dimensionally homogeneous.

Solving for $d\theta$ and using the square brackets to indicate dimensional equations for the physical quantities, we get

$$[d\theta] = \left[\frac{1}{x\sec^2(\theta)} \times \left(\frac{dy}{dt}\right) \times dt\right]$$

$$= \frac{1}{[L][L^0M^0T^0]^2} \times \frac{[L]}{[T]} \times [T]$$

$$= [L^0M^0T^0]$$

Thus, $d\theta$ is a pure number with no dimensions in L(ength), M(ass) and T(ime). Only constants and ratios of two physical quantities of the same nature have dimensionless form. Dimensional equation $[L^0M^0T^0]$ of $d\theta$ indicates that $d\theta$ cannot be in degrees but only in radians which by definition is a ratio of two lengths. Had $d\theta$ been in degrees we would have seen a quantity with units in degrees on the RHS of (1) due to requirement of dimensional homogeneity. Thus units of $\frac{d\theta}{dt}$ are rad/s.

Now,

$$r^2 = x^2 + y^2$$

Therefore,

$$2r\left(\frac{dr}{dt}\right) = 0 + 2y\left(\frac{dy}{dt}\right) \text{ since } x \text{ is a constant}$$

$$\frac{dr}{dt} = \frac{y}{r}\left(\frac{dy}{dt}\right) \qquad \textbf{(2)}$$

Units of $\frac{dr}{dt}$ are same as those of $\frac{dy}{dt}$ which are m/s. $\frac{y}{r}$ reduce to dimensionless form.