

Q Find $I = \int \sec^5 x \, dx$

[Problem: Integration by parts]

Solution

$$\begin{aligned}
 I &= \int \underbrace{\sec^3 x}_u \underbrace{\sec^2 x \, dx}_{dv} \\
 \Rightarrow I &= \underbrace{\sec^3 x}_u \underbrace{\tan x}_v - \int \underbrace{\tan x}_v \underbrace{3 \sec^2 x \sec x \tan x \, dx}_{du} \\
 \Rightarrow I &= \sec^3 x \tan x - \int \tan x \, 3 \sec^3 x \tan x \, dx \\
 \Rightarrow I &= \sec^3 x \tan x - 3 \int \tan^2 x \sec^3 x \, dx \\
 \Rightarrow I &= \sec^3 x \tan x - 3 \int (\sec^2 x - 1) \sec^3 x \, dx \\
 \Rightarrow I &= \sec^3 x \tan x - 3 \int (\sec^5 x - \sec^3 x) \, dx \\
 \Rightarrow I &= \sec^3 x \tan x - \underbrace{3 \int \sec^5 x \, dx}_{3I} + 3 \int \sec^3 x \, dx \\
 \Rightarrow 4I &= \sec^3 x \tan x + \underbrace{3 \int \sec^3 x \, dx}_{I_2} \\
 \Rightarrow 4I &= \sec^3 x \tan x + 3 \int \underbrace{\sec x}_u \underbrace{\sec^2 x \, dx}_{dv} \\
 \Rightarrow 4I &= \sec^3 x \tan x + 3 \left(\underbrace{\sec x \tan x}_u \underbrace{\sec x}_v - \int \underbrace{\tan x}_v \underbrace{\sec x \tan x \, dx}_{du} \right) \\
 \Rightarrow 4I &= \sec^3 x \tan x + 3 \sec x \tan x - 3 \int \tan^2 x \sec x \, dx \\
 \Rightarrow 4I &= \sec^3 x \tan x + 3 \sec x \tan x - 3 \int (\sec^2 x - 1) \sec x \, dx \\
 \Rightarrow 4I &= \sec^3 x \tan x + 3 \sec x \tan x - \underbrace{3 \int \sec^3 x \, dx}_{I_2 \text{ Again}} + 3 \int \sec x \, dx \\
 \Rightarrow 4I &= \sec^3 x \tan x + 3 \sec x \tan x + 3 \ln(\sec x + \tan x) - \underbrace{3 \int \sec^3 x \, dx}_{\text{See (1)}} \\
 \Rightarrow 4I &= \sec^3 x \tan x + 3 \sec x \tan x + 3 \ln(\sec x + \tan x) + \sec^3 x \tan x - 4I \\
 \Rightarrow 8I &= 2 \sec^3 x \tan x + 3 \sec x \tan x + 3 \ln(\sec x + \tan x) \\
 \Rightarrow I &= \frac{1}{4} \sec^3 x \tan x + \frac{3}{8} \sec x \tan x + \frac{3}{8} \ln(\sec x + \tan x) \\
 \Rightarrow I &= \frac{1}{4} \frac{\sin x}{\cos^4 x} + \frac{3}{8} \frac{\sin x}{\cos^2 x} + \frac{3}{8} \ln(\sec x + \tan x)
 \end{aligned}$$

recall:

$$\frac{d}{dx} \tan(x) = \sec^2 x$$

$$\frac{d}{dx} \sec(x) = \sec x \tan x$$

(1)

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Answer