

## Exercise

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1. It is required to distribute 1000 one-dollar bills among certain number of envelopes such that any whole amount up to 1000 dollars can be paid simply by giving appropriate envelopes whose enclosed amounts add to required payment. The payments are to be made to families subject to following rules:
  - 1) No payment is made to family that is absent at the payment time
  - 2) Wife receives two envelopes of any given value
  - 3) Husband receives only one envelope of any given value
  - 4) Only wife and husband are allowed to receive the payments. Each family receives amount less than or equal to \$1000.

Answer the following questions.

- a) How many envelopes are needed to make payment of any whole amount up to \$1000
- b) If the family receives a payment of \$728, what amount is payable individually to husband and to wife?
- c) If the family receives a payment of \$1000, what amount is payable individually to husband and to wife?

### **Solution**

#### **(a) Calculating number of envelopes**

We do not know what amounts to enclose in the envelopes and we do not know how many envelopes are needed. To make a payment of \$1000, if we give two envelopes of \$300 to wife and one envelope of \$400 to husband we followed only two rules but the requirement in (a) to be able to give any amount  $\leq$  \$1000 is not met. To be able to make any *payment*  $\leq$  \$1000 subject to three payment conditions we need to use *ternary system* to calculate the amounts to be enclosed in the envelopes.

Therefore,

$$3^0 = 1$$

$$3^1 = 3$$

$$3^2 = 9$$

$$3^3 = 27$$

$$3^4 = 81$$

$$3^5 = 243$$

$$3^6 = 729$$

$$3^7 = 2187$$

Since the maximum payment is limited to \$1000, we do not need to consider amounts greater than \$729. Therefore, seven distinct payment denominations of 1, 3, 9, 27, 81, 243 and 729 are sufficient to generate any whole dollar amount  $\leq$  \$1000

The maximum envelope would be required when wife receives all payments and the trits would be [2,2,2,2,2,2,2]

$$\begin{aligned} &\text{Payment corresponding to trits [2,2,2,2,2,2,2]} \\ &= 1 \times 2 + 3 \times 2 + 9 \times 2 + 27 \times 2 + 81 \times 2 + 243 \times 2 + 729 \times 2 \\ &= 2186 \end{aligned}$$

Clearly the \$2186 > \$1000 and hence number of envelopes required would be < 14. The next logical possibility of trits is [2,2,2,2,2,2,1] when wife receives 6 distinct payments in 12 envelopes and husband receives 1 payment.

$$\begin{aligned} &\text{Payment corresponding to trits [2,2,2,2,2,2,1]} \\ &= 1 \times 2 + 3 \times 2 + 9 \times 2 + 27 \times 2 + 81 \times 2 + 243 \times 2 + 729 \times 1 \\ &= 1457 \end{aligned}$$

since \$1457 > \$1000, number of envelopes required would be < 13.

The next logical possibility of trits is [2,2,2,2,2,2,0] when wife receives 6 distinct payments in 12 envelopes and husband receives 0 payment.

$$\begin{aligned} &\text{Payment corresponding to trits [2,2,2,2,2,2,0]} \\ &= 1 \times 2 + 3 \times 2 + 9 \times 2 + 27 \times 2 + 81 \times 2 + 243 \times 2 + 729 \times 0 \\ &= 728 \end{aligned}$$

since \$728 > \$1000, number of envelopes required would be  $\leq$  12.

Therefore, number of envelopes required to pay any required amount  $\leq$  12

### **b) Payment case \$728**

This has been done before.

	\$1	\$3	\$9	\$27	\$81	\$243	\$729	Total
# of env.to wife	2	2	2	2	2	2	0	
# of env.to husband	0	0	0	0	0	0	0	
Total \$	2	6	18	54	162	486	0	728
Total amount paid to family in 12 envelopes =								728

In above, all payments rules are followed and wife received all the envelopes in duplicate. The family received total amount \$728.

### c) Payment case \$1000

The trits[1000]=[1,0,0,1,0,1,1]

Since all trits are 1s the payments go to husband only.

	\$1	\$3	\$9	\$27	\$81	\$243	\$729	Total
# of env.to wife	0	0	0	0	0	0	0	
# of env.to husband	1	0	0	1	0	1	1	
Total \$	1	0	0	27	0	243	729	1000
Number of envelopes required=								4

Whole amount is payable only to husband. The payments follow all ternary rules.

### Table of various payable amounts

Bold numbers indicate amounts payable to wife in two envelopes of equal value

Amount (\$)	\$1	\$3	\$9	\$27	\$81	\$243	\$729	Total
997	1	<b>6</b>	<b>18</b>	0	0	243	729	997
991	1	0	<b>18</b>	0	0	243	729	991
983	<b>2</b>	0	9	0	0	243	729	983
977	<b>2</b>	3	0	0	0	243	729	977
971	<b>2</b>	<b>6</b>	<b>18</b>	<b>54</b>	<b>162</b>	0	729	971
967	1	3	<b>18</b>	<b>54</b>	<b>162</b>	0	729	967
953	<b>2</b>	<b>6</b>	0	<b>54</b>	<b>162</b>	0	729	953
947	<b>2</b>	0	0	<b>54</b>	<b>162</b>	0	729	947
941	2	3	<b>18</b>	27	<b>162</b>	0	729	941
937	1	0	<b>18</b>	27	<b>162</b>	0	729	937