Example

Show that $\lim_{n \to \infty} \frac{2^n}{n!} = 0$

Proof

$$2^n = (1+x)^n \text{ where } x = 1$$

$$= {}^nC_0 (x)^0 + {}^nC_1 (x)^1 + {}^nC_2 (x)^2 + \ldots + {}^nC_r (x)^r + \ldots + {}^nC_n x^n$$

$$= \sum_{r=0}^{n} x^r \, {}^nC_r$$

$$= \sum_{r=0}^{n} x^r \left(\frac{n!}{r!(n-r)!} \right)$$

$$= n! \sum_{r=0}^{n} x^r \left(\frac{1}{r!(n-r)!} \right)$$

Now substitute back $x = 1$

$$\frac{2^n}{n!} = \sum_{r=0}^{n} (1)^r \left(\frac{1}{r!(n-r)!} \right)$$

Taking limit as $n \to \infty$

$$\lim_{n \to \infty} \frac{2^n}{n!} = 0$$

Following two cases are added to illustrate why above solution works. Above solution was posted earlier in AP Forum and has not changed.

Case I: $n = 3$

Find $\lim_{n \to 3} \frac{2^n}{n!}$

From above solution

$$\frac{2^n}{n!} = \sum_{r=0}^{n}(1)^r\left(\frac{1}{r!(n-r)!}\right)$$

Note that for $n=3$, only $n+1$ binomial coefficents are expected.

Taking limit as $n \to 3$ and $0 \le r \le 3$

$$\lim_{n\to 3}\frac{2^n}{n!} =$$

$$\lim_{n\to 3}\frac{1}{0!(n-0)!} + \lim_{n\to 3}\frac{1}{1!(n-1)!} + \lim_{n\to 3}\frac{1}{2!(n-2)!} + \lim_{n\to 3}\frac{1}{3!(n-3)!}$$

$$= \frac{1}{0!(3-0)!} + \frac{1}{1!(3-1)!} + \frac{1}{2!(3-2)!} + \frac{1}{3!(3-3)!}$$

$$= \frac{4}{3}$$

Case II: $n=\infty$

Find $\lim_{n\to\infty}\frac{2^n}{n!}$

From above solution

$$\frac{2^n}{n!} = \sum_{r=0}^{n}(1)^r\left(\frac{1}{r!(n-r)!}\right)$$

Note that for $n=\infty$, $(\infty+1)$ binomial coefficents are expected.

Taking limit as $n \to \infty$ and $0 \le r \le \infty$

$$\lim_{n\to\infty} \frac{2^n}{n!} =$$

$$\lim_{n\to\infty} \frac{1}{0!(n-0)!} + \lim_{n\to\infty} \frac{1}{1!(n-1)!} + \lim_{n\to\infty} \frac{1}{2!(n-2)!} +$$

$$\lim_{n\to\infty} \frac{1}{3!(n-3)!} + \ldots + \lim_{n\to\infty} \frac{1}{n!(n-n)!}$$

$$= \frac{1}{0!(\infty-0)!} + \frac{1}{1!(\infty-1)!} + \frac{1}{2!(\infty-2)!} + \frac{1}{3!(\infty-3)!} + \ldots + \frac{1}{\infty!(0)!}$$

$= 0$ (Recall that $0! = 1$)